Homework

October 28, 2019

1 Lecture 2

Useful definitions. Consider set $Q \subseteq \mathbb{R}^n$ and a point $x_0 \in Q$. Tangent cone for Q at x is the set

$$T(x_0, Q) := \{s \in \mathbb{R}^n : \exists \alpha(\lambda) = o(\lambda) \in \mathbb{R}^n \text{ s.t. } x_0 + \lambda s + \alpha(\lambda) \in Q, \ \forall \lambda \text{ sufficiently small}\}.$$

If Q is convex, $T(x_0, Q)$ coincides with the cone of feasible directions

$$T(x_0, Q) = \overline{\{\lambda(x - x_0) : x \in Q, \ \lambda \geq 0\}}.$$

Given a cone K, the conjugate cone is

$$K^* := \{s \in \mathbb{R}^n : s^T x \geq 0, \ \forall x \in K\}.$$

If the set Q is convex, then $-T^*(x_0, Q)$ is called normal cone for Q at x_0

$$N(x_0, Q) := \{s \in \mathbb{R}^n : s^T(x - x_0) \leq 0, \ \forall x \in Q\}.$$

Optimality conditions for minimization problem

$$\min_{x \in Q} f(x)$$

- If f is general differentiable, Q is general, and x^* is local minimum, then $\langle \nabla f(x^*), s \rangle \geq 0$ for all $s \in T(x^*, Q)$. (Equivalently, $\nabla f(x^*) \in T^*(x^*, Q)$).
- If f is general differentiable, Q is convex, and x^* is local minimum, then $\langle \nabla f(x^*), x - x^* \rangle \geq 0$ for all $x \in Q$. (Equivalently, $-\nabla f(x^*) \in N(x^*, Q)$). If f is convex, this is also a sufficient condition.
- If f is general convex, Q is convex, and x^* is local (and global) minimum,

  $$0 \in \partial(f(x^*) + \delta(x^*, Q)) = \partial f(x^*) + N(x^*, Q).$$

  (Equivalently, there exists $g^* \in \partial f(x^*)$ such that $\langle g^*, x - x^* \rangle \geq 0$ for all $x \in Q$). Since f is convex, this is also a sufficient condition.

1. Solve the optimization problem using appropriate optimality conditions listed above

$$\min_{x\in\mathbb{R}^n}\{\|x-a\|_2 : c^T x \le b\},$$

where $a, c, b \in \mathbb{R}^n$ are given parameters. Consider cases $c^T a \le b$ and $c^T a > b$. Hint: is the problem convex? Is the objective differentiable? Which optimality condition should be used in this case.

2. Solve the optimization problem using appropriate optimality conditions listed above

$$\min_{(x,y)\in\mathbb{R}^2}\{3x+2y : \sqrt{|x|}+\sqrt{|y|} \le 2,\ x \le 0,\ y \ge 0\}.$$

Hint: is the problem convex? Is the objective differentiable? Which optimality condition should be used in this case.

3. Solve the optimization problem using the Lagrange function and KKT conditions.

$$\min_{(x,y)\in\mathbb{R}^2}\{(x-1)^2+(y+1)^2 : y \ge |x|,\ 3y+x=4\}.$$

4. Find the Lagrange dual for the problem

$$\min_{X\in\mathbb{R}^{n\times n}_+}\left\{\sum_{i,j=1}^{n}(C_{ij}X_{ij}+X_{ij}\ln X_{ij}) :\ X1_n = a,\ X^T 1_n = b\right\},$$

where $C \in \mathbb{R}^{n\times n}_+$ is a given matrix, $a, b \in \mathbb{R}^n$ are given vectors and $1_n \in \mathbb{R}^n$ denotes a vector with all components equal to one. Consider $X \in \mathbb{R}^{n\times n}_+$ as a constraint given by set $Q = \mathbb{R}^{n\times n}_+$ rather than a system of inequality constraints.

5. Find the Lagrange dual for the problem

$$\min_{x\in\mathbb{R}^n}\left\{c^T x : Ax = b,\ x \ge 0\right\},$$

where $A \in \mathbb{R}^{m\times n}$ is a given matrix, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are given vectors and inequality $x \ge 0$ is componentwise.